

MAXIMAL SUBSET OF PAIRWISE NON-COMMUTING ELEMENTS OF FINITE MINIMAL NON-ABELIAN GROUPS

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ABSTRACT. Let G be a group. A subset X of G is a set of pairwise non-commuting elements if $xy \neq yx$ for any two distinct elements x and y in X . If $|X| \geq |Y|$ for any other set of pairwise non-commuting elements Y in G , then X is said to be a maximal subset of pairwise non-commuting elements. In this paper we determine the cardinality of a maximal subset of pairwise non-commuting elements for finite minimal non-abelian groups.

1. INTRODUCTION

Let G be a non-abelian group and let X be a maximal subset of pairwise non-commuting elements of G . The cardinality of such a subset is denoted by $\omega(G)$. Also $\omega(G)$ is the maximal clique size in the non-commuting graph of a group G . Let $Z(G)$ be the center of G . The non-commuting graph of a group G is a graph with $G \setminus Z(G)$ as the vertices and join two distinct vertices x and y , whenever $xy \neq yx$. By a famous result of B. H. Neumann [12], answering a question of P. Erdős, the finiteness of $\omega(G)$ in G is equivalent to the finiteness of the factor group $G/Z(G)$. Pyber [13] has shown that there is some constant c such that $|G : Z(G)| \leq c^{\omega(G)}$. Moreover various attempts have been made to find $\omega(G)$ for some groups G , see for example [1], [2], [3], [7], [8] and [9].

In this paper we find $\omega(G)$ for any finite minimal non-abelian group. A minimal non-abelian group is a non-abelian group such that all its proper subgroups are abelian. A useful structure of these groups is given in [10, Aufgaben III. 5.14], which states that the order of a minimal non-abelian group G has at most two distinct prime divisors and if G is not a p -group, then only one of its sylow subgroup is

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normal in G . Following [5, Lemma 116.1 (a)], we see that $\omega(G) = p + 1$ for any finite minimal non-abelian p -group G . Therefore in this paper we assume that G is a finite minimal non-abelian group, which is not a p -group and we show that $\omega(G) = |Q| + 1$, where Q is the normal q -Sylow subgroup of G .

Throughout this paper we use the following notation. p denotes a prime number. $\mathcal{C}_G(x)$ is the centralizer of an element x in a group G . A group G is called an AC -group if the centralizer of every non-central element of G is abelian.

2. MAIN RESULT

First we state two following lemmas that are needed for the main result of this paper.

Lemma 2.1. *The following conditions on a group G are equivalent.*

- (i) G is an AC -group.
- (ii) If $[x, y] = 1$ then $\mathcal{C}_G(x) = \mathcal{C}_G(y)$, where $x, y \in G \setminus Z(G)$.

Proof. This is straightforward. See also [14, Lemma 3.2]. □

Lemma 2.2. [3, Lemma 2.3] *Let G be an AC -group.*

- (i) *If $a, b \in G \setminus Z(G)$ with distinct centralizers, then $\mathcal{C}_G(a) \cap \mathcal{C}_G(b) = Z(G)$.*
- (ii) *If $G = \cup_{i=1}^k \mathcal{C}_G(a_i)$, where $\mathcal{C}_G(a_i)$ and $\mathcal{C}_G(a_j)$ are distinct proper subgroups of G for $1 \leq i < j \leq k$, then $\{a_1 \dots a_k\}$ is a maximal set of pairwise non-commuting elements in G .*

Now we find $\omega(G)$, for a finite minimal non-abelian group G in which G is not a p -group.

The following theorem gives a structure for finite minimal non-abelian groups which play an important role in our proof of the main Theorem.

Theorem 2.3. [10, Aufgaben III. 5.14]. *Let G be a finite minimal non-abelian group. Then*

- (i) *the order of G has at most two distinct prime divisors,*

- (ii) if $|G|$ is not a power of a prime then $G = PQ$, where P is a cyclic p -Sylow subgroup of G and Q is the elementary abelian minimal normal q -Sylow subgroup of G .

Lemma 2.4. *Let G be a finite minimal non-abelian group and $G = PQ$, where P is a cyclic p -Sylow subgroup of G and Q is the elementary abelian minimal normal q -Sylow subgroup of G . Then*

- (i) $G' = Q$,
- (ii) $G' \cap Z(G) = 1$ and so $Z(G)$ is a p -subgroup of G ,
- (iii) $\mathcal{C}_G(P) = \mathcal{N}_G(P) = P$,
- (iv) $\mathcal{C}_G(b) = Z(G) \times Q$ for any $1 \neq b \in Q$.

Proof. (i) $G' \leq Q$ since $G/Q \cong P$. Now the result follows from the fact that Q is minimal normal subgroup of G .

(ii) We have $G' \cap Z(G) = Q \cap Z(G)$ is normal in G and if $Q \leq Z(G)$, then G is abelian which is impossible. This yields that $Q \cap Z(G) = 1$ and so $Z(G)$ is a p -subgroup of G .

(iii) If $P \not\leq \mathcal{N}_G(P)$, then there exists $x \in \mathcal{N}_G(P)$ of order q . Hence $x \in Q$. Let $P = \langle a \rangle$, then $[a, x] \in P$ and so $[a, x] = 1$ by (i). This implies that $x \in Z(G)$. Therefore $x = 1$ by (ii) and so $\mathcal{N}_G(P) = P$. The rest is obvious.

(iv) If $1 \neq b \in Q$, then by (ii), $b \in Q \setminus Z(G)$ and so $\mathcal{C}_G(b) \not\leq G$ is abelian. Since $Q \leq \mathcal{C}_G(b)$, we may write $\mathcal{C}_G(b) \cong Q \times P_0$, where P_0 is the p -Sylow subgroup of $\mathcal{C}_G(b)$. Therefore $[P_0, Q] = 1$. Moreover $P_0 \leq P^g$ for some $g \in G$ and $G = P^g Q$, which implies that $P_0 \leq Z(G)$. Furthermore $Z(G) \leq P_0$ by (ii) and the fact that $Z(G) \leq \mathcal{C}_G(b)$. This complete the proof. \square

Theorem 2.5. *Let G be a finite minimal non-abelian group and $G = PQ$, where P is a cyclic p -Sylow subgroup of G and Q is the elementary abelian minimal normal q -Sylow subgroup of G . Then $\omega(G) = |Q| + 1$.*

Proof. Let $|G| = p^\alpha q^\beta$ and $P_1 = P, P_2, \dots, P_m$ be all distinct p -Sylow subgroups of G and $P_i = \langle a_i \rangle$ for $1 \leq i \leq m$. Obviously $m = q^\beta$ and $\mathcal{C}_G(a_i) = P_i$ by Lemma 2.4(iii) and so $a_i \notin Z(G)$ since $G = P_i Q$. Now let $1 \neq b \in Q$, then $b \notin Z(G)$ by Lemma 2.4(ii). Moreover for $1 \leq i \leq m$ we have $\mathcal{C}_G(a_i) \neq \mathcal{C}_G(b)$, for otherwise we see that $b \in Z(G)$, which is a contradiction. Now we calculate the

order of $A = \mathcal{C}_G(a_1) \cup \cdots \cup \mathcal{C}_G(a_m) \cup \mathcal{C}_G(b)$. By Lemma 2.2(i) and the fact that G is an AC-group, we see that $|A| = \sum_{i=1}^m (|\mathcal{C}_G(a_i)| - |Z(G)|) + |\mathcal{C}_G(b)|$. Moreover by Lemma 2.4(iii), (iv) we have $|\mathcal{C}_G(a_i)| = |P_i| = p^\alpha$ and $|\mathcal{C}_G(b)| = |Z(G)|q^\beta$. Therefore $|A| = |G|$. This yields that $G = \mathcal{C}_G(a_1) \cup \cdots \cup \mathcal{C}_G(a_m) \cup \mathcal{C}_G(b)$, and so $\omega(G) = |Q| + 1$ by Lemma 2.2(ii). \square

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